AL
P MATHS
Paper II

2010
Beacon College

高考純數模擬考試 8 卷二

This paper must be answered in English

3 hours
By Oscar Tam

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C) answer book.
3. Answer any FOUR questions in Section B, using the AL(E) answer book.
4. Unless otherwise specified, all working must be clearly shown.

Not to be taken away before the end of the examination session

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.

(1) (a) Find $\lim_{x\to\infty}(4x+3)\cos\dfrac{\pi x}{2x+3}$.

(b) Find $\lim_{x\to 0}\dfrac{\int_0^x \sin(t^3)\,dt}{x^4}$.

(6 marks)

(2) Let $f(x)=\tan^{-1}x$, $x\in\mathbf{R}$.

(a) Show that

$$(1+x^2)f''(x)+2xf'(x)=0.$$

(b) (i) Show that for any positive integer n,

$$(1+x^2)f^{(n+2)}(x)+2(n+1)xf^{(n+1)}(x)+n(n+1)f^{(n)}(x)=0.$$

(ii) Find $f^{(7)}(0)$ and $f^{(8)}(0)$.

(7 marks)

(3) Let $f(x)=\begin{cases} x^4\cos\dfrac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$.

(a) For $x\neq 0$, evaluate $f'(x)$ and $f''(x)$.

(b) Show that $f''(0)$ exists.

(c) Is $f''(x)$ continuous at $x=0$?

(7 marks)

(4) (a) Evaluate $\displaystyle\int \frac{x\,dx}{1-\dfrac{1}{4^x+1}}$.

(b) Hence, or otherwise, evaluate $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{k}{n^2\left(1-\dfrac{1}{4^{\frac{k}{n}}+1}\right)}$.

(6 marks)

(5) (a) Resolve $\dfrac{5x^2-8x-16}{x^2(x-2)}$ into partial fractions.

(b) Using the substitution $u=2+e^x$, or otherwise, evaluate $\displaystyle\int\frac{(2-e^x)^2(3+e^x)}{(2+e^x)^2}dx$.

(c) Let D be the region bounded by the curve $y = \dfrac{\left(2e^{-x}-1\right)\sqrt{3e^{-x}+1}}{e^{-\frac{x}{2}}\left(2e^{-x}+1\right)}$, the positive x-axis and the positive y-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(7 marks)

(6) Consider the curve $H : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $x > 0$, a and b are positive constants. $P(x_1, y_1)$ is a variable point on H and Q is the point $(0, c)$, where $c \neq 0$.

(a) Express the length of PQ in terms of a, b, c and y_1.

(b) If P' is a point on H such that $P'Q$ represents the shortest distance from Q to H.

(i) Find the coordinates of P'.

(ii) Is $P'Q$ the normal to H at P'? Explain briefly.

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.

(7) Let $f(x)=\dfrac{3-2|x|}{e^{\frac{1}{|x|}}}$, $x \neq 0$.

(a) For $x>0$, find $f'(x)$ and show that $f''(x)=\dfrac{3-8x}{x^4 e^{\frac{1}{x}}}$.

(2 marks)

(b) For $x>0$, find the range of values of x such that

(i) $f'(x)>0$,

(ii) $f'(x)<0$,

(iii) $f''(x)>0$,

(iv) $f''(x)<0$.

(4 marks)

(c) Find all relative extreme point(s) and point(s) of inflexion of $y=f(x)$.

(3 marks)

(d) Find the asymptote(s) of the graph of $y=f(x)$.

(4 marks)

(e) Sketch the graph of $y=f(x)$.

(2 marks)

(8) (a) Let $f(x)$ and $g(x)$ be two real functions continuous on the interval $[a,b]$. By considering the integral of the function $[\lambda f(x)+g(x)]^2$ on $[a,b]$, set up a quadratic inequality in the parameter λ. Hence, show that

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \left\{\int_a^b [f(x)]^2 dx\right\}\left\{\int_a^b [g(x)]^2 dx\right\}.$$

(4 marks)

(b) Let $h(x)$ be a function with continuous derivative on $[0,1]$ satisfying $h(0)=0$ and $h(1)=0$.

(i) Show that $h(x)=\int_0^x h'(t)dt = -\int_x^1 h'(t)dt$ for any $x\in[0,1]$.

(ii) Using (a), or otherwise, show that for $x\in[0,1]$,

$$[h(x)]^2 \le x\int_0^x [h'(t)]^2 dt .$$

Hence, deduce that for $x\in\left[0,\frac{1}{2}\right]$,

$$[h(x)]^2 \le x\int_0^{\frac{1}{2}} [h'(t)]^2 dt .$$

(iii) Show that for $x\in\left[\frac{1}{2},1\right]$,

$$[h(x)]^2 \le (1-x)\int_{\frac{1}{2}}^1 [h'(t)]^2 dt .$$

(iv) Show that

$$\int_0^1 [h(x)]^2\,dx \le \frac{1}{8}\int_0^1 [h'(x)]^2\,dx .$$

(11 marks)

(9) (a) For any positive integer n and $t \in (-1, 1)$, show that

$$\frac{1}{1+t} = 1 - t + t^2 - \Lambda + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t} .$$

Hence, deduce that for any $x \in (-1, 1)$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \Lambda + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-1)^n t^n}{1+t}\,dt$$

and

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \Lambda - \frac{x^n}{n} - \int_0^x \frac{t^n}{1-t}\,dt .$$

(7 marks)

(b) Using (a), or otherwise, show that for any $x \in (0, 1)$, and for any positive integer k,

$$0 \le \ln\left(\frac{1+x}{1-x}\right) - 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \Lambda + \frac{x^{2k+1}}{2k+1}\right) \le \frac{2}{1-x^2}\left(\frac{x^{2k+3}}{2k+3}\right).$$

(4 marks)

(c) Using (b), or otherwise, show that

$$\lim_{k\to\infty}\left[\frac{1}{4} + \frac{1}{3}\left(\frac{1}{4}\right)^3 + \frac{1}{5}\left(\frac{1}{4}\right)^5 + \Lambda + \frac{1}{2k+1}\left(\frac{1}{4}\right)^{2k+1}\right] = \frac{1}{2}\ln\frac{5}{3} .$$

(4 marks)

(10) Consider the curve $P:\begin{cases} x = at^2 \\ y = 2at \end{cases}$, where $t \in \mathbf{R}$.

Let $A(at_1{}^2, 2at_1)$ and $B(at_2{}^2, 2at_2)$ be two distinct points on P, where t_1 and t_2 are non-zero real numbers.

(a) (i) Find the equation of the normal to P at A.

(ii) Prove that AB is normal to P at A if and only if $t_1{}^2 + t_1 t_2 + 2 = 0$.

(6 marks)

(b) Let $F = (a, 0)$. It is given that the chord of P joining A and F cuts P again at C.

(i) Find the y-coordinate of C in terms of a and t_1.

(ii) Suppose D is the point of intersection of the tangents at A and B. Also, AB is normal to P at A.

(A) Show that CD is horizontal.

(B) Show that the mid-point of AD lies on the line $x = -a$.

(C) Find the equation of the locus of D at t_1 varies.

(9 marks)

(11) (a) Let $f(x)$ be a twice differentiable function defined on $(0,\infty)$ where $f''(x)\le 0$ for $x>0$.

(i) Let *a* and *b* be two distinct positive real numbers. Let *c* lies on the open interval with end points *a* and *b*. Using the mean value theorem, show that

$$(c-a)f'(c)\le f(c)-f(a)$$

and

$$(c-b)f'(c)\le f(c)-f(b).$$

(ii) For any positive real numbers *r*, *a* and *b*, show that

$$f\left(\frac{ar+b}{1+r}\right)\ge\frac{rf(a)+f(b)}{1+r}.$$

(9 marks)

(b) Let $x_1, x_2, \cdots, x_n$ be distinct positive real numbers.

(i) Using (a), show that for all positive integers $n\ge 2$,

$$\ln\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)\ge\frac{\ln x_1+\ln x_2+\cdots+\ln x_n}{n}.$$

(ii) Show that for all positive integers $n\ge 2$,

$$\frac{x_1+x_2+\cdots+x_n}{n}\ge\sqrt[n]{x_1x_2\cdots x_n}.$$

(6 marks)

End of paper

Marking Scheme

Marks

(1) (a) $\lim_{x\to\infty}(4x+3)\cos\frac{\pi x}{2x+3} = \lim_{x\to\infty}\frac{\cos\frac{\pi x}{2x+3}}{\frac{1}{4x+3}}$

$$= \lim_{x\to\infty}\frac{-\sin\frac{\pi x}{2x+3}\left[\frac{(2x+3)\pi-\pi x(2)}{(2x+3)^2}\right]}{-\frac{4}{(4x+3)^2}}$$ 1M

(L'Hospital's rule)

$$= \lim_{x\to\infty}\frac{3\pi(4x+3)^2}{4(2x+3)^2}\sin\frac{\pi x}{2x+3}$$

$$= \lim_{x\to\infty}\frac{3\pi\left(4+\frac{3}{x}\right)^2}{4\left(2+\frac{3}{x}\right)^2}\sin\frac{\pi}{2+\frac{3}{x}}$$ 1M

$$= \frac{3\pi}{4}\cdot\frac{4^2}{2^2}\sin\frac{\pi}{2}$$

$$= 3\pi.$$ 1A

(b) $\lim_{x\to 0}\frac{\int_0^x \sin(t^3)\,dt}{x^4} = \lim_{x\to 0}\frac{\sin x^3}{4x^3}$ (L'Hospital's rule) 1M

$$= \lim_{y\to 0}\frac{\sin y}{4y}$$ (let $y = x^3$) 1M

$$= \frac{1}{4}\cdot 1$$

$$= \frac{1}{4}.$$ 1A

(2) (a) $f'(x) = \frac{1}{1+x^2}$

and

$$f''(x) = -\frac{2x}{(1+x^2)^2}.$$

Thus,

$$(1+x^2)f''(x)+2xf'(x) = (1+x^2)\left[-\frac{2x}{(1+x^2)^2}\right]+2x\cdot\frac{1}{1+x^2}$$
$$=0.$$ 1

(b) (i) Using the Leibniz's rule,

$$(1+x^2)f^{(n+2)}(x)+C_1^n(2x)f^{(n+1)}(x)+C_2^n(2)f^{(n)}(x)+2\left[xf^{(n+1)}(x)+C_1^n f^{(n)}(x)\right]=0$$ 1M
$$(1+x^2)f^{(n+2)}(x)+2(n+1)xf^{(n+1)}(x)+[n(n-1)+2n]f^{(n)}(x)=0$$
$$(1+x^2)f^{(n+2)}(x)+2(n+1)xf^{(n+1)}(x)+n(n+1)f^{(n)}(x)=0.$$ 1

(ii) Put $x=0$ into (b) (i),

$$f^{(n+2)}(0)=-n(n+1)f^{(n)}(0).$$ 1A

Thus,

$$\begin{aligned} f^{(7)}(0) &= -5(6)f^{(5)}(0) \\ &= [-5(6)][-3(4)]f^{(3)}(0) \\ &= [-5(6)][-3(4)][-1(2)]f'(0) \\ &= -720 \end{aligned}$$
1M, 1A

and

$$\begin{aligned} f^{(8)}(0) &= -6(7)f^{(6)}(0) \\ &= [-6(7)][-4(5)]f^{(4)}(0) \\ &= [-6(7)][-4(5)][-2(3)]f''(0) \\ &= 0. \end{aligned}$$
1A

(3) (a) For $x\neq 0$,

$$\begin{aligned} f'(x) &= 4x^3\cos\frac{1}{x}+x^4\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) \\ &= 4x^3\cos\frac{1}{x}+x^2\sin\frac{1}{x} \end{aligned}$$
1A

and

$$\begin{aligned} f''(x) &= 12x^2\cos\frac{1}{x}+4x^3\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)+2x\sin\frac{1}{x}+x^2\cos\frac{1}{x}\left(-\frac{1}{x^2}\right) \\ &= 12x^2\cos\frac{1}{x}+6x\sin\frac{1}{x}-\cos\frac{1}{x}. \end{aligned}$$
1A

(b) Consider

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0}\frac{x^4\cos\frac{1}{x}-0}{x-0}\\ &= \lim_{x\to 0}x^3\cos\frac{1}{x}\\ &= 0\end{aligned}$$

as $\left|\cos\frac{1}{x}\right|\le 1$ and $\lim_{x\to 0}x^3=0$.

Thus, $f'(0)=0$. 1A

Consider

$$\begin{aligned}\lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0} &= \lim_{x\to 0}\frac{4x^3\cos\frac{1}{x}+x^2\sin\frac{1}{x}-0}{x-0}\\ &= \lim_{x\to 0}\left(4x^2\cos\frac{1}{x}+x\sin\frac{1}{x}\right).\end{aligned}$$

As $\left|\cos\frac{1}{x}\right|\le 1$ and $\lim_{x\to 0}4x^2=0$, $\lim_{x\to 0}4x^2\cos\frac{1}{x}=0$. 1M

Also, as $\left|\sin\frac{1}{x}\right|\le 1$ and $\lim_{x\to 0}x=0$, $\lim_{x\to 0}x\sin\frac{1}{x}=0$.

Thus,

$$\begin{aligned}\lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0} &= \lim_{x\to 0}4x^2\cos\frac{1}{x}+\lim_{x\to 0}x\sin\frac{1}{x}\\ &= 0+0\\ &= 0.\end{aligned}$$

Hence, $f''(0)$ exists and equals 0. 1A

(c) Consider

$$\lim_{x\to 0}f''(x) = \lim_{x\to 0}\left(12x^2\cos\frac{1}{x}+6x\sin\frac{1}{x}-\cos\frac{1}{x}\right),$$ 1M

which does not exist.

Hence, $f''(x)$ is not continuous at $x=0$. 1A

(4) (a)

$$\int \frac{x\,dx}{1-\frac{1}{4^x+1}} = \int \frac{x(4^x+1)dx}{4^x}$$

$$= \int x\,dx + \int \frac{x}{4^x}dx \qquad \text{1A}$$

$$= \frac{x^2}{2} + \int \frac{x}{4^x}dx .$$

Let $u = x$, $dv = 4^{-x}dx$.

$du = dx$, $v = -\frac{4^{-x}}{\ln 4}$.

$$\int \frac{x}{4^x}dx = -\frac{x4^{-x}}{\ln 4} + \int \frac{4^{-x}}{\ln 4}dx$$

$$= -\frac{x}{(\ln 4)4^x} - \frac{4^{-x}}{(\ln 4)^2} + C . \qquad \text{1A}$$

Thus,

$$\int \frac{x\,dx}{1-\frac{1}{4^x+1}} = \frac{x^2}{2} - \frac{x}{(\ln 4)4^x} - \frac{1}{4^x(\ln 4)^2} + C . \qquad \text{1A}$$

(b)

$$\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{k}{n^2\left(1-\frac{1}{4^{\frac{k}{n}}+1}\right)} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{2n}\frac{\frac{k}{n}}{1-\frac{1}{4^{\frac{k}{n}}+1}} \qquad \text{1A}$$

$$= \int_0^2 \frac{x}{1-\frac{1}{4^x+1}}dx \qquad \text{1A}$$

$$= \left[\frac{x^2}{2} - \frac{x}{(\ln 4)4^x} - \frac{1}{4^x(\ln 4)^2}\right]_0^2$$

$$= 2 - \frac{1}{8\ln 4} + \frac{15}{16(\ln 4)^2} . \qquad \text{1A}$$

(5) (a) Let

$$\frac{5x^2-8x-16}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$= \frac{(A+C)x^2+(B-2A)x-2B}{x^2(x-2)} .$$

$$\therefore \begin{cases} A+C=5 \\ B-2A=-8 \\ -2B=-16 \end{cases} \Rightarrow \begin{cases} A=8 \\ B=8 \\ C=-3 \end{cases}.$$

$$\therefore \frac{5x^2-8x-16}{x^2(x-2)} = \frac{8}{x}+\frac{8}{x^2}-\frac{3}{x-2}.$$ 1A

(b) Let $u=2+e^x$, $du=e^x dx$.

$$\int \frac{(2-e^x)^2(3+e^x)}{(2+e^x)^2}dx = \int \frac{[2-(u-2)]^2[3+(u-2)]}{u^2}\cdot\frac{du}{u-2}$$

$$= \int \frac{(4-u)^2(1+u)}{u^2(u-2)}du$$ 1A

$$= \int \left[1-\frac{5u^2-8u-16}{u^2(u-2)}\right]du$$

$$= \int \left(1-\frac{8}{u}-\frac{8}{u^2}+\frac{3}{u-2}\right)du$$ 1A

$$= u-8\ln|u|+\frac{8}{u}+3\ln|u-2|+C$$

$$= e^x-8\ln(e^x+2)+\frac{8}{e^x+2}+3x+C.$$ 1A

(c) Required volume $= \pi\int_0^{\ln 2}\left[\frac{(2e^{-x}-1)\sqrt{3e^{-x}+1}}{e^{-\frac{x}{2}}(2e^{-x}+1)}\right]^2 dx$ 1M + 1A

$$= \pi\int_0^{\ln 2}\frac{(2-e^x)^2(3+e^x)}{(2+e^x)^2}dx$$

$$= \pi\left[e^x-8\ln(e^x+2)+\frac{8}{e^x+2}+3x\right]_0^{\ln 2}$$

$$= \frac{\pi}{3}-13\pi\ln 2+8\pi\ln 3.$$ 1A

(6) (a) $PQ = \sqrt{(x_1-0)^2+(y_1-c)^2}$

$$= \sqrt{a^2\left(1+\frac{y_1{}^2}{b^2}\right)+(y_1-c)^2}$$

$$= \sqrt{\left(\frac{a^2+b^2}{b^2}\right)y_1{}^2-2cy_1+(a^2+c^2)}.$$ 1A

(b) (i) Let $f(y_1)=\left(\frac{a^2+b^2}{b^2}\right)y_1^{\ 2}-2cy_1+\left(a^2+c^2\right)$.

As $\frac{a^2+b^2}{b^2}>0$, $f(y_1)$ is the least when

$$\begin{aligned} y_1 &= -\frac{(-2c)}{2\left(\frac{a^2+b^2}{b^2}\right)} \\ &= \frac{b^2c}{a^2+b^2}. \end{aligned}$$ 1A

Thus,

$$\begin{aligned} x_1^{\ 2} &= a^2\left(1+\frac{y_1^{\ 2}}{b^2}\right) \\ &= a^2\left[1+\frac{\left(\frac{b^2c}{a^2+b^2}\right)^2}{b^2}\right] \\ &= \frac{a^2\left[\left(a^2+b^2\right)^2+b^2c^2\right]}{\left(a^2+b^2\right)^2}. \end{aligned}$$

$$\therefore\ P'=\left(\frac{a}{a^2+b^2}\sqrt{\left(a^2+b^2\right)^2+b^2c^2},\frac{b^2c}{a^2+b^2}\right).$$ 1A

(ii) Consider

$$\begin{aligned} \text{Slope of } P'Q &= \frac{\frac{b^2c}{a^2+b^2}-c}{\frac{a}{a^2+b^2}\sqrt{\left(a^2+b^2\right)^2+b^2c^2}-0} \\ &= -\frac{ac}{\sqrt{\left(a^2+b^2\right)^2+b^2c^2}}. \end{aligned}$$ 1A

Also,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}.$$

Thus,

Slope of tangent to H at P' $=\dfrac{b^2\left[\dfrac{a}{a^2+b^2}\sqrt{\left(a^2+b^2\right)^2+b^2c^2}\right]}{a^2\left(\dfrac{b^2c}{a^2+b^2}\right)}$ 1M

$$=\frac{\sqrt{\left(a^2+b^2\right)^2+b^2c^2}}{ac}.$$

As

(Slope of $P'Q$) $\times$ (Slope of tangent to H at P') $=-1$, 1M

$P'Q$ is the normal to H at P'. 1

(7) (a) For $x>0$, $f(x)=\dfrac{3-2x}{e^{\frac{1}{x}}}$.

Thus,

$$f'(x)=\frac{e^{\frac{1}{x}}(-2)-(3-2x)e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{\left(e^{\frac{1}{x}}\right)^2}$$

$=\dfrac{-2x^2-2x+3}{x^2e^{\frac{1}{x}}}$ 1A

and

$$f''(x)=\frac{\left(x^2e^{\frac{1}{x}}\right)(-4x-2)-\left(-2x^2-2x+3\right)\left[2xe^{\frac{1}{x}}+x^2e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)\right]}{x^4\left(e^{\frac{1}{x}}\right)^2}$$

$$=\frac{x^2(-4x-2)-\left(-2x^2-2x+3\right)(2x-1)}{x^4e^{\frac{1}{x}}}$$

$= \dfrac{3-8x}{x^4 e^{\frac{1}{x}}}$. 1A

(b) (i) $f'(x) > 0 \Leftrightarrow 0 < x < \dfrac{\sqrt{7}-1}{2}$. 1A

(ii) $f'(x) < 0 \Leftrightarrow x > \dfrac{\sqrt{7}-1}{2}$. 1A

(iii) $f''(x) > 0 \Leftrightarrow 0 < x < \dfrac{3}{8}$. 1A

(iv) $f''(x) < 0 \Leftrightarrow x > \dfrac{3}{8}$. 1A

(c) Refer to the tables.

x	$\left(0, \dfrac{\sqrt{7}-1}{2}\right)$	$\dfrac{\sqrt{7}-1}{2}$	$\left(\dfrac{\sqrt{7}-1}{2}, \infty\right)$
$f'(x)$	+	0	–
$f(x)$	↑	0.40	↓

x	$\left(0, \dfrac{3}{8}\right)$	$\dfrac{3}{8}$	$\left(\dfrac{3}{8}, \infty\right)$
$f''(x)$	+	0	–
$f(x)$		0.16	

Note that $f(x)$ is even, the graph of $y = f(x)$ is symmetrical about the y-axis. 1M

Thus, $\left(\dfrac{\sqrt{7}-1}{2}, 0.40\right)$ and $\left(\dfrac{1-\sqrt{7}}{2}, 0.40\right)$ are relative maximum points. Also, $\left(\dfrac{3}{8}, 0.16\right)$ and $\left(-\dfrac{3}{8}, 0.16\right)$ are points of inflexion. 1A 1A

(d) Consider

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{3-2x}{e^{\frac{1}{x}}} = 0$$

and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{3+2x}{e^{-\frac{1}{x}}} = 0.$$

$\therefore$ $x=0$ is not a vertical asymptote of the graph. 1

Consider

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{3-2x}{xe^{\frac{1}{x}}} = \lim_{x\to\infty}\left(\frac{3}{x}-2\right)e^{-\frac{1}{x}} = -2$$

and

$$\lim_{x\to\infty}[f(x)-(-2)x] = \lim_{x\to\infty}\left(\frac{3-2x}{e^{\frac{1}{x}}}+2x\right) = \lim_{x\to\infty}\left\{\frac{3}{e^{\frac{1}{x}}}+\frac{2}{e^{\frac{1}{x}}}\left[x\left(e^{\frac{1}{x}}-1\right)\right]\right\}.$$

As

$$\lim_{x\to\infty} x\left(e^{\frac{1}{x}}-1\right) = \lim_{x\to\infty}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}} = \lim_{x\to\infty}\frac{e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \text{ (L'Hospital's rule)} = \lim_{x\to\infty} e^{\frac{1}{x}} = 1.$$

1M

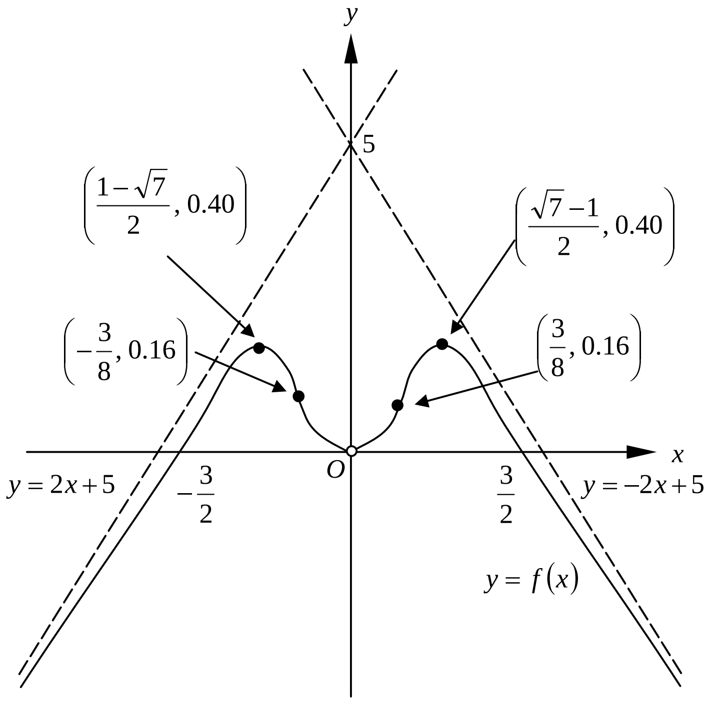
Thus,

$$\lim_{x\to\infty}[f(x)-(-2)x] = 3+2(1) = 5.$$

$\therefore$ $y=-2x+5$ is an oblique asymptote as $x\to\infty$. 1A

By symmetry, $y = 2x + 5$ is an oblique asymptote as $x \to -\infty$. | 1A

(d)



1A for the relative minimum point and the point of inflexion

1A for the shape of the curve

(8) (a) For $x \in [a, b]$,

$$[\lambda f(x) + g(x)]^2 \geq 0$$

$$\int_a^b [\lambda f(x) + g(x)]^2 dx \geq 0 \qquad \text{1M}$$

$$\lambda^2 \int_a^b [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x)dx + \int_a^b [g(x)]^2 dx \geq 0.$$

<u>Case 1: If $f(x) = 0$ for all $a \leq x \leq b$</u>

In this case, the result is trivial. | 1A

<u>Case 2: If $f(x) \neq 0$ for some $a \leq x \leq b$</u>

By the continuity of *f*, $\int_a^b [f(x)]^2 dx > 0$. Thus,

$$\left[2\int_a^b f(x)g(x)dx\right]^2 - 4\left\{\int_a^b [f(x)]^2 dx\right\}\left\{\int_a^b [g(x)]^2 dx\right\} \leq 0 \qquad \text{1M}$$

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \left\{\int_a^b [f(x)]^2 dx\right\}\left\{\int_a^b [g(x)]^2 dx\right\}. \qquad \text{1}$$

(b) (i) Consider

$$\int_0^x h'(t)dt = h(t)\Big|_0^x = h(x) - h(0) = h(x)$$ 1

and

$$-\int_x^1 h'(t)dt = -h(t)\Big|_x^1 = -[h(1) - h(x)] = h(x).$$ 1

(ii) Let $f(x) \equiv 1$ and $g(x) = h'(x)$ in (a), we have 1M

$$\left[\int_0^x h'(t)dt\right]^2 \le \left(\int_0^x dt\right)\left\{\int_0^x [h'(t)]^2 dt\right\}$$

$$[h(x)]^2 \le x\int_0^x [h'(t)]^2 dt.$$ 1

For $x \in \left[0, \frac{1}{2}\right]$, we have

$$\int_0^x [h'(t)]^2 dt \le \int_0^{\frac{1}{2}} [h'(t)]^2 dt$$ 1M

as $[h'(t)]^2 \ge 0$.

Thus,

$$[h(x)]^2 \le x\int_0^{\frac{1}{2}} [h'(t)]^2 dt.$$ 1

(iii) Let $f(x) \equiv 1$ and $g(x) = h'(x)$ in (a), we have

$$\left[\int_x^1 h'(t)dt\right]^2 \le \left(\int_x^1 dt\right)\left\{\int_x^1 [h'(t)]^2 dt\right\}$$

$$[h(x)]^2 \le (1-x)\int_x^1 [h'(t)]^2 dt.$$

For $x \in \left[\frac{1}{2}, 1\right]$, we have

$$\int_x^1 [h'(t)]^2 dt \le \int_{\frac{1}{2}}^1 [h'(t)]^2 dt$$

as $[h'(t)]^2 \geq 0$.

Thus,

$$[h(x)]^2 \leq (1-x)\int_{\frac{1}{2}}^{1}[h'(t)]^2\,dt\,.$$ 1

(iv) Using (b) (ii) and (b) (iii),

$$\begin{aligned}\int_0^{\frac{1}{2}}[h(x)]^2\,dx &\leq \left(\int_0^{\frac{1}{2}} x\,dx\right)\left\{\int_0^{\frac{1}{2}}[h'(t)]^2\,dt\right\} \\ &= \left.\frac{x^2}{2}\right|_0^{\frac{1}{2}} \cdot \left\{\int_0^{\frac{1}{2}}[h'(t)]^2\,dt\right\} \\ &= \frac{1}{8}\int_0^{\frac{1}{2}}[h'(t)]^2\,dt \quad \text{.......... (1)}\end{aligned}$$

1M

1A

and

$$\begin{aligned}\int_{\frac{1}{2}}^{1}[h(x)]^2\,dx &\leq \left[\int_{\frac{1}{2}}^{1}(1-x)\,dx\right]\left\{\int_{\frac{1}{2}}^{1}[h'(t)]^2\,dt\right\} \\ &= \left.\left(x-\frac{x^2}{2}\right)\right|_{\frac{1}{2}}^{1} \cdot \left\{\int_{\frac{1}{2}}^{1}[h'(t)]^2\,dt\right\} \\ &= \frac{1}{8}\int_{\frac{1}{2}}^{1}[h'(t)]^2\,dt\,. \quad \text{.......... (2)}\end{aligned}$$

1A

$(1)+(2)$:

$$\begin{aligned}\int_0^1[h(x)]^2\,dx &\leq \frac{1}{8}\int_0^{\frac{1}{2}}[h'(t)]^2\,dt + \frac{1}{8}\int_{\frac{1}{2}}^{1}[h'(t)]^2\,dt \\ &= \frac{1}{8}\int_0^1[h'(x)]^2\,dx\,.\end{aligned}$$

1

(9) (a) Consider

$$\begin{aligned}1-t+t^2-\Lambda+(-1)^{n-1}t^{n-1} &= \frac{1-(-t)^n}{1-(-t)} \\ &= \frac{1-(-1)^n t^n}{1+t}\,.\end{aligned}$$

1M

Hence,

$$\frac{1}{1+t}=1-t+t^2-\Lambda+(-1)^{n-1}t^{n-1}+\frac{(-1)^n t^n}{1+t}.$$ 1

For any $x\in(-1,1)$,

$$\int_0^x \frac{1}{1+t}dt=\int_0^x\left[1-t+t^2-\Lambda+(-1)^{n-1}t^{n-1}+\frac{(-1)^n t^n}{1+t}\right]dt$$ 1M

$$\ln(1+t)\Big|_0^x=\left[t-\frac{t^2}{2}+\frac{t^3}{3}-\Lambda+\frac{(-1)^{n-1}t^n}{n}\right]_0^x+\int_0^x\frac{(-1)^n t^n}{1+t}dt$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\Lambda+\frac{(-1)^{n-1}x^n}{n}+\int_0^x\frac{(-1)^n t^n}{1+t}dt.$$ 1

By replacing x by $-x$, we have 1M

$$\begin{aligned}\ln(1-x)&=-x-\frac{(-x)^2}{2}+\frac{(-x)^3}{3}-\Lambda+\frac{(-1)^{n-1}(-x)^n}{n}+\int_0^{-x}\frac{(-1)^n t^n}{1+t}dt\\&=-x-\frac{x^2}{2}-\frac{x^3}{3}-\Lambda-\frac{x^n}{n}+\int_0^{-x}\frac{(-1)^n t^n}{1+t}dt.\end{aligned}$$

Note that

$$\begin{aligned}\int_0^{-x}\frac{(-1)^n t^n}{1+t}dt&=\int_0^x\frac{(-1)^n(-s)^n}{1+(-s)}(-ds)\quad(\text{let } t=-s)\\&=-\int_0^x\frac{s^n}{1-s}ds\\&=-\int_0^x\frac{t^n}{1-t}dt.\end{aligned}$$ 1M

Hence,

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\Lambda-\frac{x^n}{n}-\int_0^x\frac{t^n}{1-t}dt.$$ 1

(b) By putting $n=2k+1$ in (a), we have 1M

$$\ln(1+x)-\ln(1-x)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\Lambda+\frac{x^{2k+1}}{2k+1}\right)+\int_0^x\frac{(-1)^{2k+1}t^{2k+1}}{1+t}dt+\int_0^x\frac{t^{2k+1}}{1-t}dt$$

$$\ln\left(\frac{1+x}{1-x}\right)-2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\Lambda+\frac{x^{2k+1}}{2k+1}\right)=\int_0^x t^{2k+1}\left(\frac{1}{1-t}-\frac{1}{1+t}\right)dt.$$

Note that

$$\int_0^x t^{2k+1}\left(\frac{1}{1-t}-\frac{1}{1+t}\right)dt = 2\int_0^x \frac{t^{2k+2}}{1-t^2}\,dt.$$

For $0 \le t \le x < 1$,

$$\frac{t^{2k+2}}{1-t^2} \ge 0$$

$$\int_0^x \frac{t^{2k+2}}{1-t^2}\,dt \ge 0$$ 1M

and

$$2\int_0^x \frac{t^{2k+2}}{1-t^2}\,dt \le \frac{2}{1-x^2}\int_0^x t^{2k+2}\,dt$$ 1M

$$= \frac{2}{1-x^2}\cdot\left(\frac{x^{2k+3}}{2k+3}\right).$$

Hence,

$$0 \le \ln\left(\frac{1+x}{1-x}\right) - 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\Lambda+\frac{x^{2k+1}}{2k+1}\right) \le \frac{2}{1-x^2}\left(\frac{x^{2k+3}}{2k+3}\right).$$ 1

(c) Using (b),

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{1-x^2}\left(\frac{x^{2k+3}}{2k+3}\right) \le x+\frac{x^3}{3}+\frac{x^5}{5}+\Lambda+\frac{x^{2k+1}}{2k+1} \le \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Put $x = \frac{1}{4}$ into the above expression, 1M

$$\frac{1}{2}\ln\frac{5}{3} - \frac{16}{15}\cdot\frac{\left(\frac{1}{4}\right)^{2k+3}}{2k+3} \le \frac{1}{4}+\frac{1}{3}\left(\frac{1}{4}\right)^3+\frac{1}{5}\left(\frac{1}{4}\right)^5+\Lambda+\frac{1}{2k+1}\left(\frac{1}{4}\right)^{2k+1} \le \frac{1}{2}\ln\frac{5}{3}.$$

Note that $\lim_{k\to\infty}\frac{\left(\frac{1}{4}\right)^{2k+3}}{2k+3} = 0$, by squeezing principle, 1M + 1M

$$\lim_{k\to\infty}\left[\frac{1}{4}+\frac{1}{3}\left(\frac{1}{4}\right)^3+\frac{1}{5}\left(\frac{1}{4}\right)^5+\Lambda+\frac{1}{2k+1}\left(\frac{1}{4}\right)^{2k+1}\right] = \frac{1}{2}\ln\frac{5}{3}.$$ 1

(10) (a) (i)

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$ 1M

$$=\frac{2a}{2at}$$

$$=\frac{1}{t}.$$ 1A

Required equation:

$$\frac{y-2at_1}{x-at_1^{\ 2}}=-t_1$$

$$t_1x+y-2at_1-at_1^{\ 3}=0.$$ 1A

(ii) AB is normal to P at A

$\Leftrightarrow$ Slope of $AB=-t_1$ 1M

$\Leftrightarrow$ $\dfrac{2at_2-2at_1}{at_2^{\ 2}-at_1^{\ 2}}=-t_1$

$\Leftrightarrow$ $\dfrac{2}{t_2+t_1}=-t_1$ 1A

$\Leftrightarrow$ $t_1^{\ 2}+t_1t_2+2=0.$ 1

(b) (i) Let $C=\left(at_3^{\ 2},2at_3\right)$, where $t_3\neq t_1$.

$$\text{Slope of } AF = \text{Slope of } FC$$ 1M

$$\frac{2at_1-0}{at_1^{\ 2}-a}=\frac{2at_3-0}{at_3^{\ 2}-a}$$

$$t_1\left(t_3^{\ 2}-1\right)=t_3\left(t_1^{\ 2}-1\right)$$

$$\left(t_1t_3+1\right)\left(t_3-t_1\right)=0$$

$$t_3=-\frac{1}{t_1}.\quad \left(\Theta\ t_1\neq t_3\right)$$

Hence,

$$y\text{-coordinate of } C=2at_3$$

$$=-\frac{2a}{t_1}.$$ 1A

(ii) (A) Equation of tangent at A:

$$\frac{y-2at_1}{x-at_1^{\ 2}} = \frac{1}{t_1}$$ 1M

$$x - t_1 y + at_1^{\ 2} = 0 \text{.} \quad \cdots\cdots (1)$$

Similarly, the equation of tangent at *B*:

$$x - t_2 y + at_2^{\ 2} = 0 \text{.} \quad \cdots\cdots (2)$$

$(1)-(2)$:

$$(t_2 - t_1)y + a(t_1^{\ 2} - t_2^{\ 2}) = 0$$
$$y = a(t_1 + t_2) \text{.}$$

Put $y = a(t_1 + t_2)$ into (1),

$$x - at_1(t_1 + t_2) + at_1^{\ 2} = 0$$
$$x = at_1 t_2 \text{.}$$

$\therefore\ D = (at_1 t_2, a(t_1 + t_2))$. 1A

Using (a) (ii), as $t_1^{\ 2} + t_1 t_2 + 2 = 0$,

$$a(t_1 + t_2) = a\left(-\frac{2}{t_1}\right)$$ 1M
$$= -\frac{2a}{t_1} \text{.}$$

Hence, the *y*-coordinates of *C* and *D* are equal.

$\therefore$ *CD* is a horizontal line. 1

(B) *x*-coordinate of the mid-point of *AD* $= \dfrac{at_1^{\ 2} + at_1 t_2}{2}$

$$= \frac{a}{2}(t_1^{\ 2} + t_1 t_2)$$
$$= \frac{a}{2}(-2)$$
$$= -a \text{.}$$ 1A

Hence, the mid-point of *AD* lies on the line $x = -a$. 1

(C) Let $D = (x, y)$.

$$\therefore \begin{cases} x = at_1t_2 \\ y = a(t_1 + t_2) \end{cases}.$$

Note that $t_2 = -\dfrac{t_1^{\ 2}+2}{t_1}$, we have

$$\begin{aligned} x &= at_1\left(-\frac{t_1^{\ 2}+2}{t_1}\right) \\ &= -a(t_1^{\ 2}+2) \end{aligned}$$

and

$$\begin{aligned} y &= a\left(t_1 - \frac{t_1^{\ 2}+2}{t_1}\right) \\ &= -\frac{2a}{t_1}. \end{aligned}$$

Thus,

$$\begin{aligned} x &= -a\left[\left(-\frac{2a}{y}\right)^2 + 2\right] \\ y^2(2a+x)+4a^3 &= 0. \end{aligned}$$ 1A

(11) (a) (i) <u>Case 1: $a < c < b$</u>

Using the mean value theorem, there exist $\xi_1 \in (a,c)$ and $\xi_2 \in (c,b)$ such that

$$\frac{f(c)-f(a)}{c-a} = f'(\xi_1) \text{ and } \frac{f(b)-f(c)}{b-c} = f'(\xi_2).$$ 1A

As $f''(x) \le 0$, $f'(x)$ is decreasing for $x > 0$. 1M

$$a < \xi_1 < c < \xi_2 < b$$
$$f'(a) \ge f'(\xi_1) \ge f'(c) \ge f'(\xi_2) \ge f'(b)$$ 1M
$$f'(a) \ge \frac{f(c)-f(a)}{c-a} \ge f'(c) \ge \frac{f(b)-f(c)}{b-c} \ge f'(b).$$

Thus,

$$(c-a)f'(c) \le f(c)-f(a)$$ 1

and

$$\begin{aligned}(b-c)f'(c) &\geq f(b)-f(c)\\(c-b)f'(c) &\leq f(c)-f(b).\end{aligned}$$ 1

<u>Case 2: $b<c<a$</u>

This is similar to case 1.

(ii) <u>Case 1: $a \neq b$</u>

When $a<b$,

$$\begin{aligned}ar+a &< ar+b\\a &< \frac{ar+b}{1+r}\end{aligned}$$

and

$$\begin{aligned}ar+b &< b+br\\\frac{ar+b}{1+r} &< b.\end{aligned}$$

Hence, $a<\frac{ar+b}{1+r}<b$. 1A

When $b<a$,

$$\begin{aligned}ar+b &< a+ar\\\frac{ar+b}{1+r} &< a\end{aligned}$$

and

$$\begin{aligned}b+br &< ar+b\\b &< \frac{ar+b}{1+r}.\end{aligned}$$

Hence, $b<\frac{ar+b}{1+r}<a$.

Using (a) (i), put $c=\frac{ar+b}{1+r}$, we have 1M

$$\left(\frac{ar+b}{1+r}-a\right)f'\left(\frac{ar+b}{1+r}\right)\leq f\left(\frac{ar+b}{1+r}\right)-f(a) \quad \text{.......... (1)}$$

and

$$\left(\frac{ar+b}{1+r}-b\right) f'\left(\frac{ar+b}{1+r}\right) \le f\left(\frac{ar+b}{1+r}\right)-f(b) . \quad \cdots\cdots (2)$$

$r\times(1)+(2)$: 1M

$$r\left(\frac{ar+b}{1+r}-a\right) f'\left(\frac{ar+b}{1+r}\right)+\left(\frac{ar+b}{1+r}-b\right) f'\left(\frac{ar+b}{1+r}\right)$$
$$\le rf\left(\frac{ar+b}{1+r}\right)-rf(a)+f\left(\frac{ar+b}{1+r}\right)-f(b)$$
$$0 \le (r+1)f\left(\frac{ar+b}{1+r}\right)-rf(a)-f(b)$$
$$f\left(\frac{ar+b}{1+r}\right) \ge \frac{rf(a)+f(b)}{1+r} .$$
1

Case 2: $a=b$

The result is trivial in this case.

(b) (i) Let $f(x)=\ln x$, $x>0$. 1M

Note that $f''(x)=-\frac{1}{x^2}\le 0$ for $x>0$. 1

Using (a) (ii), by putting $r=1$, $a=x_1$, $b=x_2$, we have 1M

$$f\left(\frac{x_1+x_2}{2}\right) \ge \frac{f(x_1)+f(x_2)}{2}$$
$$\ln\left(\frac{x_1+x_2}{2}\right) \ge \frac{\ln x_1+\ln x_2}{2} .$$

Therefore, the statement is true when $n=2$.

Assume the statement is true when $n=k$, i.e.,

$$\ln\left(\frac{x_1+x_2+\Lambda+x_k}{k}\right) \ge \frac{\ln x_1+\ln x_2+\Lambda+\ln x_k}{k} .$$

Consider the statement when $n=k+1$, by putting
$a=\frac{x_1+x_2+\Lambda+x_k}{k}$, $b=x_{k+1}$ and $r=k$ in (a) (ii), 1M

$$\ln\left(\frac{\frac{x_1+x_2+\Lambda+x_k}{k}\cdot k+x_{k+1}}{1+k}\right)\geq\frac{k\ln\left(\frac{x_1+x_2+\Lambda+x_k}{k}\right)+\ln x_{k+1}}{1+k}$$

$$\ln\left(\frac{x_1+x_2+\Lambda+x_{k+1}}{k+1}\right)\geq\frac{k\left(\frac{\ln x_1+\ln x_2+\Lambda+\ln x_k}{k}\right)+\ln x_{k+1}}{1+k}$$

$$\ln\left(\frac{x_1+x_2+\Lambda+x_{k+1}}{k+1}\right)\geq\frac{\ln x_1+\ln x_2+\Lambda+\ln x_k+\ln x_{k+1}}{k+1}.$$

Therefore, the statement is also true when $n=k+1$.

By M.I., the statement is true for all positive integers *n*.

(ii) By (b) (i),

$$\ln\left(\frac{x_1+x_2+\Lambda+x_n}{n}\right)\geq\frac{\ln(x_1x_2\Lambda\ x_n)}{n}$$ 1M

$$\ln\left(\frac{x_1+x_2+\Lambda+x_n}{n}\right)\geq\ln\sqrt[n]{x_1x_2\Lambda\ x_n}$$

$$\frac{x_1+x_2+\Lambda+x_n}{n}\geq\sqrt[n]{x_1x_2\Lambda\ x_n}.$$ 1